Finite-Temperature Field Theory on the Light Front[*]

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Abstract.

The formulation of statistical physics using light-front quantization, instead of conventional equal-time boundary conditions, has important advantages for describing relativistic statistical systems, such as heavy ion collisions. We develop light-front field theory at finite temperature and density with special attention to quantum chromodynamics. First, we construct the most general form of the statistical operator allowed by the Poincaré algebra. In light-front quantization, the Green's functions of a quark in a medium can be defined in terms of just 2-component spinors and does not lead to doublers in the transverse directions. Since the theory is non-local along the light cone, we use causality arguments to construct a solution to the related zero-mode problem. A seminal property of light-front Green's functions is that they are related to parton densities in coordinate space. Namely, the diagonal and off-diagonal parton distributions measured in hard scattering experiments can be interpreted as light-front density matrices.

Dirac's front form of relativistic dynamics [1] has remarkable advantages in high energy and nuclear physics. Most appealing is the simplicity of the vacuum (the ground state of the free theory is also the ground state of the full theory) and the existence of boost-invariant light-cone wavefunctions (see Ref. [2] for a review.) This makes light-front quantization a natural candidate for the description of systems for which boost invariance is an issue, such as the fireball created in a heavy ion collision or the small-x features of a nuclear wavefunction. Until now, however, most applications of Dirac's front form refer to the case of zero temperature. It is clearly important to exploit the advantages of light-front quantization also for thermal field theory. In this talk, we report our recent findings, see Ref. [3], where we applied front-form dynamics to statistical

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physics and investigated the prospects and challenges of this approach for quantum chromodynamic systems. Valuable work in this direction has already been done by several authors [4, 5, 6].

The form of the statistical operator $\widehat{w}$ at finite temperature and density can be obtained from very general considerations. Our result for $\widehat{w}$, is compatible with the findings of [4, 5], *i.e.* $\widehat{w}$ is always the exponential of the equal time energy $\widehat{P}^0$ in the local rest frame of the system. Our derivation follows Ref. [7].

The light-front Liouville theorem [3] ($\mathrm{i}\partial^-\widehat{w} = [\widehat{P}^-, \widehat{w}]$) requires that in equilibrium, $\widehat{w}$ is a function of only those Poincaré generators which commute with the light-front Hamiltonian $\widehat{P}^-$. In addition, since systems far apart from each other must be uncorrelated, the density operator of the combined system has to factorize into the density operators of the subsystems. Consequently, in equilibrium $\ln(\widehat{w})$ must be a linear combination of the additive constants of motion, namely the four components of the momentum $\widehat{P}^\mu$ and the 3-component of the angular momentum vector $\boldsymbol{J}$. Hence,

$$\ln(\widehat{w}) = \alpha - \beta \left(u_\nu \widehat{P}^\nu - \omega \widehat{J}_3 - \sum_l \mu_l \widehat{Q}_l \right). \tag{1}$$

Here, β is the inverse temperature and u_ν is the four velocity of the system, cf. Ref. [5]. In addition, ω is the angular velocity at which the body rotates. Additional conserved charges $\widehat{Q}_l$ are included along with their chemical potentials μ_l. In quantum mechanics, of course, one can simultaneously specify only charges which commute with each other, *e.g.* one cannot specify all four P_ν for systems with nonzero angular momentum.

We remark that T is the same as the instant-form temperature, but the chemical potential has a different meaning. On the light-front, densities are given by +-components of currents, and not by 0-components. This is essential for the proper generalization of parton distributions (PDFs) to finite temperature, since the latter are also defined as +-components. Finite temperature PDFs are useful for parton recombination models (see *e.g.* [8]), even though they cannot be measured in deep inelastic scattering.

We choose $\alpha = 0$ as normalization in Eq. (1), so that the partition function is given by $\mathcal{Z} = \mathrm{Tr}\widehat{w}$. The grand-canonical ensemble can now be written in terms of light-cone wavefunctions $\phi_{n/h}(X)$ as (let $\boldsymbol{P}_\perp = \boldsymbol{0}_\perp$),

$$\begin{aligned}
\widehat{w} &= \sum_h \sum_{n,n'} \sum_{X,X'} \exp\left\{ -\beta \left[u^+ \frac{M_h^2}{2P^+} + \frac{u^-}{2} \sum_i p_i^+ - \mu Q \right] \right\} \\
&\times \; \phi_{n/h}(X)\phi_{n'/h}^*(X')|nX\rangle\langle n'X'|.
\end{aligned} \tag{2}$$

Since the wavefunctions and the masses M_h of the eigenstates can (in principle) be obtained from discretized light-cone quantization (DLCQ) [9], this expression shows that one can also calculate all thermodynamic properties of a field theory from DLCQ.

Furthermore, since $\mathcal{Z}$ is a Lorentz scalar, all thermodynamic potentials and the entropy transform as scalars, *e.g.* the Lorentz invariant generalization of the grand-canonical potential (or of the free energy in the case of $\mu = 0$) is

$\Omega = -T \ln \mathcal{Z}(V, T, \mu)$, and the entropy is given by

$$S = -\left(\frac{\partial \Omega}{\partial T}\right)_{\mu, V} = -\frac{1}{\mathcal{Z}} \text{Tr}(\widehat{w} \ln \widehat{w}). \tag{3}$$

The role of the total energy of the system is now played by the expectation value of $u_\nu \widehat{P}^\nu$, $U = \langle u_\nu \widehat{P}^\nu \rangle$. As usual, $\Omega = U - TS - \mu Q$. All known relations between thermodynamic potentials remain valid.

The quantities β, u_ν, ω and μ_l, have the meaning of Lagrange multipliers that hold the mean values of the constants of motion fixed, while entropy is maximized. In an ideal gas for example, the maximum entropy is attained for occupation numbers given by Fermi-Dirac and Bose-Einstein statistics [4, 5],

$$n(u_\nu p^\nu) = \frac{g}{e^{\beta(u_\nu p^\nu - \mu)} \pm 1} - \frac{g}{e^{\beta(u_\nu p^\nu + \mu)} \pm 1}, \tag{4}$$

assuming that particles carry charge $+1$ and antiparticles charge -1. (Note that $u_\nu p^\nu \geq 0$.) Here, p^ν is the 4-momentum of a single particle and the degeneracy factor for different spin states is denoted by g. The Lagrange multipliers define the equilibrium conditions for two systems. In complete equilibrium with each other, both systems must have the same values of temperature, u_ν, ω and μ_l, *i.e.* no internal motion of macroscopic parts of the system is possible in equilibrium (at least in the absence of vortex lines [7].)

The simplicity of the light-front vacuum, usually considered an advantage, seems to bear problems as far as phase transitions are concerned. However, the statistical weight of a configuration is maximized for minimal equal-time energy rather than for minimal light-front energy. Therefore, the ground state, *i.e.* the state the system is in at $T = 0$, is in general different from the light-front vacuum. For that reason, the authors of Ref. [5] obtain the standard pattern of spontaneous symmetry breaking in ϕ^4-theory with negative mass squared. No problem arises from $1/k^+$-poles. In addition, in Ref. [4] the chiral phase transition in the Nambu–Jona-Lasinio Model on the light-front is reproduced. We conclude that this approach is poised for the study of phase transitions in more complicated field theories, such as QCD.

Until now one could get the impression that thermodynamics and statistical physics on the light front are identical to the usual instant form approach, except for a trivial change of variables. That this is not the case becomes most clear, when one studies fermions on the light-front. In light-front field theory, the Dirac equations can be written as a set of two coupled equations for 2-component spinors, see *e.g.* appendix of Ref. [3]. Only one of these equations contains a time derivative, the other one is a constraint. As a consequence, the entire theory can be formulated in terms of 2-component spinors, very much like a non-relativistic theory.

The time-ordered Green's functions of a fermion in a medium is defined in terms of the Heisenberg operators of the dynamical field components [5, 10],

$$iG_{\alpha, \beta}(r_1, r_2) = \langle \widehat{\psi}_\alpha(r_1) \widehat{\psi}_\beta^\dagger(r_2) \rangle \Theta(r_1^+ - r_2^+) - \langle \widehat{\psi}_\beta^\dagger(r_2) \widehat{\psi}_\alpha(r_1) \rangle \Theta(r_2^+ - r_1^+), \tag{5}$$

where the average $\langle \ldots \rangle$ is to be taken with the appropriate ensemble, and $\alpha, \beta \in \{1, 2\}$. This definition of the Green's function includes the case of zero

temperature. Therefore, the conventional light-front quantization at temperature $T = 0$ can be formulated in terms of $G_{\alpha,\beta}$ as well. The Green's function is the fundamental object of this approach. In addition, the retarded (R) and advanced (A) Green's functions are defined by the anticommutators

$$\mathrm{i}G_{\alpha,\beta}^{R,A}(r_1,r_2) \;\; = \;\; \pm\langle\{\widehat{\psi}_\alpha(r_1),\widehat{\psi}_\beta^\dagger(r_2)\}\rangle\Theta(\pm(r_1^+ - r_2^+)), \qquad (6)$$

where the upper sign refers to $G_{\alpha,\beta}^R$ and the lower sign to $G_{\alpha,\beta}^A$. We remark, that in a gauge theory, it is also necessary to include a (path ordered) gauge link along the light-cone by redefining the fermion fields, see Ref. [3].

The Green's functions of a fermion in an ideal gas of temperature $T = 1/\beta$ were presented first in Ref. [5]. In momentum space, adjusted to our notation, they read

$$\widetilde{G}_{\alpha,\beta}^{(0)R,A}(k) \;\; = \;\; \delta_{\alpha,\beta}\frac{k^+}{k^2 - m^2 \pm \mathrm{i}\epsilon\mathrm{sgn}(uk)}, \qquad (7)$$

$$\widetilde{G}_{\alpha,\beta}^{(0)}(k) \;\; = \;\; \delta_{\alpha,\beta}\left(\mathrm{P}\frac{k^+}{k^2 - m^2} - \mathrm{isgn}(uk)\pi\tanh\left(\frac{uk}{2T}\right)k^+\delta(k^2 - m^2)\right), (8)$$

where P refers to principle value prescription. The pole prescriptions $\pm\mathrm{i}\epsilon\mathrm{sgn}(uk)$ for the retarded and advanced Green's functions are another manifestation of the special meaning of the equal-time energy. These prescriptions ensure that $G_{\alpha,\beta}^{(0)R}(r)$ vanishes outside the forward lightcone, while $G_{\alpha,\beta}^{(0)A}(r)$ is non-vanishing only inside the backward lightcone. Most importantly, knowledge of the correct pole prescription eliminates ambiguities in the definition of the non-local operator $1/k^+$, which appears in the free light-cone Hamiltonian. However, the correct prescription for the $1/k^+$-pole depends on the type of Green's function and on the value of the other momentum components.

Another remarkable property of the light-front Green's functions is, that if the theory is discretized on a lattice in coordinate space, the factor k^+ in the numerator leads to only one pair of fermion doublers. For a transverse lattice lattice approach to finite temperature $SU(\infty)$, see Ref. [11]. Moreover, it is known that no fermion doubling problem occurs in DLCQ and one can perform DLCQ calculations for a fixed value of the total charge without any sign problem.

In the limit $r^+ \to 0^\pm$, the time-ordered Green's function $G_{\alpha,\beta}$ is closely related to the one-particle density matrices for fermions and antifermions, $q_{\alpha,\beta}(\underline{r}_1,\underline{r}_2)$ and $\overline{q}_{\alpha,\beta}(\underline{r}_1,\underline{r}_2)$. In the 2-component theory, the separation of fermion and antifermion distributions requires the evaluation of a Fourier-integral of the Green's function. For quarks, one has

$$q_{\alpha,\beta}(k^+,\underline{R},\boldsymbol{r}_\perp) = -\frac{\mathrm{i}}{4\pi}\int dr^-\mathrm{e}^{+\mathrm{i}k^+r^-/2}G_{\alpha,\beta}(r_2^+ \to 0^-,\underline{r}_1,r_2^+,\underline{r}_2). \qquad (9)$$

For antiquarks, the limit $r^+ \to 0$ is taken from the other side to obtain the correct order of creation and annihilation operators [3]. Since $G_{\alpha,\beta}(r_1,r_2)$ often depends only on the difference $r = r_1 - r_2$, we introduce the variables $R = (r_1 + r_2)/2$ and $r = r_1 - r_2$. The density matrix $q_{\alpha,\beta}$ is related to the so-called Wigner function by a Fourier transform over $r_\perp$. We remark that all properties of the quantum

mechanical density matrix, such as hermiticity and positivity of the diagonal matrix elements, also apply to the light-front density matrix in $A^+ = 0$ gauge, since this gauge has only states with positive norm and no unphysical degrees of freedom. However, the light-front density matrix has matrix elements that are off-diagonal in Fock-space. The object defined in Eq. (9) is similar to the Wigner function introduced in Ref. [12].

The fermion density matrix contains all information about single-quark properties. It depends on 6 variables and is a 2×2 matrix in spinor space, which can be written as a linear combination of Pauli spin matrices. The coefficients are the density matrices for unpolarized, longitudinal, and transverse spin distributions. The diagonal matrix elements in coordinate space of $q_{\alpha,\beta}(k^+, \underline{R}, \boldsymbol{r}_\perp)$, $i.e.$ the ones with $\boldsymbol{r}_\perp = \boldsymbol{0}_\perp$, are closely related to the usual PDFs. For instance, the unpolarized collinear quark density is given by

$$q(k^+) = \frac{1}{2} \int d^3R \delta_{\alpha,\beta} q_{\beta,\alpha}(k^+, \underline{R}, \boldsymbol{r}_\perp = \boldsymbol{0}_\perp). \tag{10}$$

This parton density is normalized such that $\int_0^\infty dk^+ q(k^+) = q$, the total number of quarks in the system.

The off-diagonal matrix elements of $q_{\alpha,\beta}$ are related to generalized parton distributions (GPDs) [13] by Fourier transform [3]. The precise relation depends on the kinematics, and one has to distinguish four different domains in deeply virtual Compton scattering (DVCS). In particular, for skewedness $\zeta = 0$, GPDs can be identified as impact parameter dependent parton densities [14]. In the case of no helicity flip we find,

$$q(k^+, \boldsymbol{b}) = \int dR^- \frac{1}{2} \delta_{\beta,\alpha} q_{\alpha,\beta}(k^+, R^-, \boldsymbol{b}, \boldsymbol{r}_\perp = \boldsymbol{0}_\perp) \tag{11}$$

$$= \frac{1}{4\pi k^+} \left\langle \sum_\lambda \widehat{b}^\dagger(k^+, \boldsymbol{b}, \lambda) \widehat{b}(k^+, \boldsymbol{b}, \lambda) \right\rangle, \tag{12}$$

$$q(k^+, \boldsymbol{b}) dk^+ = \int \frac{d^2\Delta_\perp}{(2\pi)^2} e^{i\boldsymbol{b}\cdot\boldsymbol{\Delta}_\perp} H(X, \zeta = 0, t), \tag{13}$$

We use the notation of Radyushkin here [13]. The creation operator of a quark at impact parameter $\boldsymbol{b}$ is given by

$$\widehat{b}(k^+, \boldsymbol{b}, \lambda) = \int \frac{d^2k_\perp}{(2\pi)^2} e^{-i\boldsymbol{k}_\perp \cdot \boldsymbol{b}} \widehat{b}(k^+, \boldsymbol{k}_\perp, \lambda). \tag{14}$$

The destruction operator $\widehat{b}^\dagger(k^+, \boldsymbol{b}, \lambda)$ is defined analogously. We stress that for $\zeta \neq 0$, GPDs are in general not probability distributions but density matrices, which do not need to be positive.

The light-front density matrix is a natural extension of the parton model to quantum mechanics: classical parton densities are replaced by a density matrix. It would be interesting to identify other hard processes besides DVCS which are sensitive to the quantum mechanical nature of parton distributions. Most important however is the connection between the density matrix and the fermion Green's function, because that establishes a common language for high energy scattering and statistical QCD.

In summary, we have presented a new formalism for analyzing relativistic statistical systems based on light-front quantization. The new formalism provides a boost-invariant generalization of thermodynamics, and thus it has direct applicability to the QCD analysis of heavy ion collisions and other systems of relativistic particles.

References

1. P.A.M. Dirac, Rev. Mod. Phys. **21**, 392 (1949).

2. S. J. Brodsky, H. C. Pauli and S. S. Pinsky, Phys. Rept. **301**, 299 (1998).

3. J. Raufeisen and S. J. Brodsky, hep-th/0408108.

4. M. Beyer, S. Mattiello, T. Frederico and H. J. Weber, Phys. Lett. B **521**, 33 (2001); arXiv:hep-ph/0310222.

5. V. S. Alves, A. Das and S. Perez, Phys. Rev. D **66**, 125008 (2002).

6. S. Lenz, *Statistische Mechanik auf dem Lichtkegel,* Diplom Thesis, Erlangen-Nuremberg U., 1990 (unpublished, in German); S. Elser and A. C. Kalloniatis, Phys. Lett. B **375**, 285 (1996); S. J. Brodsky, Nucl. Phys. Proc. Suppl. **108**, 327 (2002); H. A. Weldon, Phys. Rev. D **67**, 085027 (2003); *ibid.* 128701 (2003); A. Das and X. x. Zhou, Phys. Rev. D **68**, 065017 (2003); A. N. Kvinikhidze and B. Blankleider, Phys. Rev. D **69**, 125005 (2004).

7. E. M. Lifshitz and L. P. Pitajewski, *Landau-Lifshitz Course of Theoretical Physics Vol. 5: Statistical Physics Part 1,* Pergamon Press, Oxford, UK, 1980.

8. R. J. Fries, arXiv:nucl-th/0403036.

9. H. C. Pauli and S. J. Brodsky, Phys. Rev. D **32**, 2001 (1985); *ibid.* 1993 (1985).

10. P. P. Srivastava and S. J. Brodsky, Phys. Rev. D **64**, 045006 (2001).

11. S. Dalley and B. van de Sande, hep-ph/0409114.

12. A. V. Belitsky, X. d. Ji and F. Yuan, Phys. Rev. D **69**, 074014 (2004).

13. D. Müller, D. Robaschik, B. Geyer, F. M. Dittes and J. Horejsi, Fortsch. Phys. **42**, 101 (1994); X. D. Ji, Phys. Rev. D **55**, 7114 (1997); Phys. Rev. Lett. **78**, 610 (1997); A. V. Radyushkin, Phys. Rev. D **56**, 5524 (1997).

14. M. Burkardt, Phys. Rev. D **62**, 071503 (2000) [Erratum-ibid. D **66**, 119903 (2002)].